Homework 10

1. (Kittel 8.3) **Photon Carnot Engine.** Consider a Carnot engine that uses as a working substance a photon gas.

(a) Given $\tau_h$ and $\tau_l$ as well as $V_1$ and $V_2$, determine $V_3$ and $V_4$.

(b) What is the heat $Q_h$ taken up and the work done by the gas during the first isothermal expansion? Are they equal to each other, as for the ideal gas?

(c) Do the two isentropic stages cancel each other, as for the ideal gas?

(d) Calculate the total work done by the gas during one cycle. Compare it with the heat taken up at $\tau_h$ and show that the energy conversion efficiency is the Carnot efficiency.

**Solution.**

(a) Consider Figure 8.6 in Kittel. The processes from 2 to 3 and from 4 to 1 are isentropic, i.e., $S$ is a constant. For photons, $S \propto VT^3$, and so

\[
V_2 T_h^3 = V_3 T_l^3
\]

\[
V_1 T_h^3 = V_4 T_l^3
\]

\[
V_3 = \left( \frac{\tau_h}{\tau_l} \right)^3 V_2
\]

\[
V_4 = \left( \frac{\tau_h}{\tau_l} \right)^3 V_1
\]

(b) $Q_h = T_h \Delta S = T_h (S_2 - S_1) = \tau_h (\sigma_2 - \sigma_1)$

\[
\sigma = \frac{4\pi^2 V}{45} \left( \frac{\tau}{\hbar c} \right)^3
\]

\[
Q_h = \frac{4\pi^2 \tau_h^4}{45\hbar^3 c^3} (V_2 - V_1) \quad (\text{Eqn 1})
\]

\[
U = \frac{\pi^2 V \tau_h^4}{15\hbar^3 c^3}
\]
\[ \Delta U = U_2 - U_1 = \frac{\pi^2 \tau_h^4}{15h^3 c^3} (V_2 - V_1) \] (Eqn 2)

\[ p = \frac{U}{3V} = \frac{\pi^2 \tau^4}{45h^3 c^3} \]

\[ W = \int_{V_i}^{V_f} p\, dV = \int_{V_i}^{V_f} \frac{\pi^2 \tau_h^4}{45h^3 c^3} \, dV = \frac{\pi^2 \tau_h^4}{45h^3 c^3} (V_2 - V_1) \] (Eqn 3)

Eqns 1, 2 and 3 give that \( Q_h = \Delta U + W \) and \( W \neq Q_h \) as in the ideal gas case.

For an ideal gas, \( U = \frac{3}{2} N k_B T \) and so \( \Delta U = U_2 - U_1 = 0 \) giving \( Q_h = W \).

For a photon gas \( U = \frac{\pi^2 V \tau^4}{15h^3 c^3} \) and so \( \Delta U = U_2 - U_1 \neq 0 \) giving \( Q_h \neq W \).

(b) From 2 to 3: \( Q = 0 \) giving \( \Delta U + W = 0 \). Hence

\[ W_{2-3} = -(U_3 - U_2) = -\frac{\pi^2}{15h^3 c^3} \left( V_3 \tau_i^4 - V_2 \tau_h^4 \right) \]

From 4 to 1: \( Q = 0 \) giving \( \Delta U + W = 0 \). Hence

\[ W_{4-1} = -(U_1 - U_4) = -\frac{\pi^2}{15h^3 c^3} \left( V_4 \tau_i^4 - V_1 \tau_h^4 \right) \]

Recall from part (a) of this question that

\[ V_2 T_h^3 = V_3 T_i^3 \]
\[ V_1 T_h^3 = V_4 T_i^3 \]

Hence

\[ W_{2-3} = -\frac{\pi^2 V \tau_h^3}{15h^3 c^3} (\tau_i - \tau_h) \]

\[ W_{4-1} = -\frac{\pi^2 V \tau_i^3}{15h^3 c^3} (\tau_h - \tau_i) \]
Summing gives
\[ W_{2-3} + W_{4-1} = \frac{\pi^2 \tau_h^3}{15 h^3 c^3} (\tau_h - \tau_i)(V_2 - V_1) \neq 0 \]

(d)
\[ W = W_{1-2} + W_{2-3} + W_{3-4} + W_{4-1} = W_{1-2} + W_{4-1} + \frac{\pi^2 \tau_h^3}{15 h^3 c^3} (\tau_h - \tau_i)(V_2 - V_1) \]

Hence
\[ W = \frac{\pi^2 \tau_h^4}{45 h^3 c^3} (V_2 - V_1) + \frac{\pi^2 \tau_h^3 \tau_i^1}{45 h^3 c^3} (V_4 - V_3) + \frac{\pi^2 \tau_h^3}{15 h^3 c^3} (\tau_h - \tau_i)(V_2 - V_1) \]

The part (a) equations
\[ V_3 = \left( \frac{\tau_h}{\tau_i} \right)^3 V_2 \]
\[ V_4 = \left( \frac{\tau_h}{\tau_i} \right)^3 V_1 \]

give
\[ W = \frac{\pi^2 \tau_h^4}{45 h^3 c^3} (V_2 - V_1) + \frac{\pi^2 \tau_h^3 \tau_i^1}{45 h^3 c^3} (V_4 - V_3) + \frac{\pi^2 \tau_h^3}{15 h^3 c^3} (\tau_h - \tau_i)(V_2 - V_1) \]

So
\[ W = \frac{\pi^2 \tau_h^3}{45 h^3 c^3} (V_2 - V_1)(\tau_h - \tau_i + 3(\tau_h - \tau_i)) = \frac{4\pi^2 \tau_h^3 (\tau_h - \tau_i)}{45 h^3 c^3} (V_2 - V_1) \]

Using Eqn 1 we get that
\[ \eta = \frac{W}{Q_h} = \frac{\tau_h - \tau_i}{\tau_h} = 1 - \frac{\tau_i}{\tau_h} \]

2. (Kittel 8.5) **Thermal Pollution.** A river with a water temperature \( T_i = 20^\circ C \) is to be used as the low temperature reservoir of a large power plant, with a steam temperature of
$T_h = 500^\circ C$. If ecological considerations limit the amount of heat that can be dumped into the river to 1500 MW, what is the largest electrical output that the plant can deliver? If improvements in hot-steam technology would permit raising $T_h$ by 100° C, what effect would this have on the plant capacity?

Solution.

$T_i = 20^\circ C = 293K$; $T_h = 500^\circ C = 773K$; $Q_2 = 1500MW$.

Note that

$$\eta = 1 - \frac{T_i}{T_h} = \frac{W}{Q_1} = \frac{W}{Q_2 + W}$$

$$\eta Q_2 + \eta W = W$$

$$W = \frac{\eta}{1 - \eta} Q_2 = \frac{1 - \frac{T_i}{T_h}}{1 - 1 + \frac{T_i}{T_h}} = \frac{T_h - T_i}{T_i} Q_2$$

Hence

$$W = \frac{773 - 293}{293}(1500MW) = 2457MW$$

If $T_h = 600^\circ C = 873K$, then we get

$$W = \frac{873 - 293}{293}(1500MW) = 2969MW$$

3. (Kittel 8.6) Room air conditioner. A room air conditioner acts as a Carnot refrigerator between an outside temperature $T_h$ and a room at a lower temperature $T_i$. The room gains heat from the outdoors at a rate $A(T_h - T_i)$; this heat is removed by the air conditioner. The power supplied to the cooling unit is $P$.

(c) Show that the steady state temperature of the room is

$$T_i = (T_h + P/2A) - \left[(T_h + P/2A)^2 - T_h^2 \right]^{1/2}$$
(d) If the outdoors is at 37°C and the room is maintained at 17°C by a cooling power of 2 kW, find the heat loss coefficient A of the room in $W K^{-1}$. A good discussion of room air conditioners is given by H. S. Leff and W. D. Teeters, Amer J. Physics 46, 19 (1978). In a realistic unit the cooling coils may be at 282 K and the outdoor heat exchanger at 378 K.

Solution.

When $Q_2 = A(T_h - T_i)$ the room will reach a steady temperature.

$$\frac{Q_2}{p} = \gamma = \frac{T_i}{T_h - T_i}$$

$$A(T_h - T_i) = \frac{T_i}{T_h - T_i} p$$

$$(T_h - T_i)^2 = T_i \frac{p}{A}$$

Hence

$$T_i^2 - T_i \left(2T_h + \frac{p}{A}\right) + T_h^2 = 0$$

Hence

$$T_i = T_h + \frac{p}{2A} \pm \sqrt{\left(T_h + \frac{p}{2A}\right)^2 - T_h^2}$$

$T_i < T_h$, and so the minus sign solution should be taken. This gives

$$T_i = T_h + \frac{p}{2A} - \sqrt{\left(T_h + \frac{p}{2A}\right)^2 - T_h^2}$$

4. (Kittel 8.9) Cooling of a nonmetallic solid to $T = 0$. We saw in Chapter 4 that the heat capacity of nonmetallic solids at sufficiently low temperatures is proportional to $T^3$, as $C = aT^3$. Assume it were possible to cool a piece of such a solid to $T = 0$ by means of a reversible refrigerator that uses the solid specimen as its (varying!) low-temperature reservoir, and for which the high-temperature reservoir has a fixed temperature $T_h$ equal
to the initial temperature $T_i$ of the solid. Find an expression for the electrical energy required.

Solution.

$$\eta = 1 - \frac{T_2}{T_i} = \frac{dW}{dQ_1}$$

$$dQ_2 = -aT_2^3dT_2$$

$$dQ_1 = dW + dQ_2 = dW - CT_2^3dT_2$$

$$1 - \frac{T_2}{T_i} = \frac{dW}{dW - CT_2^3dT_2}$$

$$\left(1 - \frac{T_2}{T_i}\right)dW - \left(1 - \frac{T_2}{T_i}\right)aT_2^3dT_2 = dW$$

$$\left(\frac{T_2}{T_i}\right)dW = -\left(1 - \frac{T_2}{T_i}\right)aT_2^3dT_2$$

$$dW = -(T_i - T_2)aT_2^2dT_2$$

$$W = -\int_{T_i}^{T_2} (T_i - T_2)aT_2^2dT_2 = \int_{T_i}^{T_2} (T_i - T_2)aT_2^2dT_2 = a\left(T_i \frac{T_2^3}{3} - \frac{T_i^4}{4}\right) = \frac{aT_i^4}{12}$$