1. Magnetic field penetration in a plate. The penetration equation may be written as \\
\[ \lambda^2 \nabla^2 B = B, \] where \( \lambda \) is the penetration depth.

(a) Show that \( B(x) \) inside a superconducting plate perpendicular to the x axis and of thickness \( \delta \) is given by

\[ B(x) = B_a \frac{\cosh(x / \lambda)}{\cosh(\delta / 2\lambda)}, \]

where \( B_a \) is the field outside the plate and parallel to it; here \( x=0 \) is at the center of the plate.

(b) The effective magnetization \( M(x) \) in the plate is defined by \( B(x) - B_a = 4\pi M(x) \).

Show that in CGS, \( 4\pi M(x) = -B_a \frac{\delta^2 - 4x^2}{8\lambda^2} \), for \( \delta \ll \lambda \).

2. Critical field of thin films.

(a) Use the result of problem 1(b), show that the free energy density at \( T=0 \)K within a superconducting film of thickness \( \delta \) in an external magnetic field \( B_a \) is given by, for \( \delta \ll \lambda \),

\[ F_s(x, B_a) = U_s(0) + \frac{(\delta^2 - 4x^2)B_a^2}{64\pi\lambda^2}. \]

(b) Show that the magnetic contribution to \( F_s \) when averaged over the thickness of the thin film is \( <F_s> = U_s(0) + \frac{(\delta / \lambda)^2 B_a^2}{96\pi} \).

(c) Show that the critical field of the thin film is proportional to \( (\lambda / \delta)H_c \), where \( H_c \) is the bulk critical field.

3. Two fluid model of a superconductor. On the two-fluid model of a superconductor we assume that at temperatures \( 0<T<T_c \) the current density may be written as the sum of the contributions of normal and superconducting electrons: \( \vec{j} = \vec{j}_N + \vec{j}_S \), where
\[ \vec{j}_N = \sigma_0 \vec{E} \] and \( \vec{j}_S \) is given by the London equation of \( \vec{j}_S = -\frac{c}{4\pi\lambda_L^2} \vec{A} \). Here \( \sigma_0 \) is an ordinary normal conductivity, decreased by the reduction in the number of normal electrons at temperature \( T \) as compared to the normal state. Neglect inertial effects on both \( \vec{j}_N \) and \( \vec{j}_S \).

(a) Show from the Maxwell equations that the dispersion relation connecting wavevector \( k \) and frequency \( \omega \) for electromagnetic waves in the superconductor is

\[ k^2 c^2 = i4\pi\sigma_0 \omega - c^2 \lambda_L^2 + \omega^2 \]

(b) If \( \tau \) is the relaxation time of the normal electrons and \( n_N \) is their concentration, show by use the expression \( \sigma_0 = n_N e^2 \tau / m \) that at frequencies \( \omega << 1/\tau \) the dispersion relation does not involve the normal electrons in an important way, so that the motion of the electrons is described by the London equation alone.

4. **London penetration depth.** (a) Take the time derivative of the London equation \( \vec{j} = -\frac{c}{4\pi\lambda_L^2} \vec{A} \) to show that \( \frac{\partial \vec{j}}{\partial t} = \frac{c^2}{4\pi\lambda_L^2} \vec{\dot{E}} \). (b) If \( m \frac{d\vec{v}}{dt} = q\vec{E} \) as for free carriers of charge \( q \) and mass \( m \), show that \( \lambda_L^2 = mc^2 / 4\pi q^2 \).

5. **Diffraction effect of Josephson junction.** Consider a junction of rectangular cross section with a magnetic field \( B \) applied in the plane of the junction, normal to an edge of with \( w \). Let the thickness of the junction be \( T \). Assume for convenience that the phase difference of the two superconductor is \( \pi/2 \) when \( B=0 \). Show that the dc current in the presence of the magnetic field is

\[ J = J_0 \frac{\sin(wTB/e\hbar)}{wTB/e\hbar} . \]