1. (Kittel 7.1) **Density of orbitals in one and two dimensions.**

   (a) Show that the density of orbitals of a free electron in one dimension is
   \[
   D_1(\varepsilon) = \frac{L}{\pi} \left( \frac{2m}{\hbar^2 \varepsilon} \right)^{1/2}
   \]
   where \( L \) is the length of the line.

   (b) Show that in two dimensions, for a square of area \( A \),
   \[
   D_2(\varepsilon) = \frac{Am}{\pi \hbar^2}
   \]
   independent of \( \varepsilon \).

2. (Kittel 7.2) **Energy of a relativistic fermi gas.** For electrons with an energy \( \varepsilon >> mc^2 \), where \( m \) is the rest mass of the electron, the energy is given by \( \varepsilon \approx pc \), where \( p \) is the momentum. For electrons in a cube of volume \( V = L^3 \) the momentum is of the form \( (\pi \hbar / L) \) multiplied by \( (n_x^2 + n_y^2 + n_z^2)^{1/2} \), exactly as for the nonrelativistic limit. (a) Show that in this extreme relativistic limit the Fermi energy of a gas of \( N \) electrons is given by
   \[
   \varepsilon_F = \hbar \pi c \left( \frac{3n}{\pi} \right)^{1/3}
   \]
   where \( n = N/V \). (b) Show that the total energy of the ground state of the gas is
   \[
   U_0 = \frac{3}{4} N \varepsilon_F
   \]
   The general problem is treated by F. Juttner, Zeitschrift fur Physik 47, 542 (1928).

3. (Kittel 7.3) **Pressure and entropy of degenerate Fermi gas.**

   (a) Show that a fermi electron gas in the ground state exerts a pressure
\[ p = \frac{(3\pi^2)^{2/3}}{5} \frac{\hbar^2}{m} \left( \frac{N}{V} \right)^{2/3} \]

In a uniform decrease of the volume of a cube every orbital has its energy raised: The energy of an orbital is proportional to \(1/L^2\) or \(1/V^{2/3}\).

(a) Find an expression for the entropy of a fermi electron gas in the region \(\tau \ll \varepsilon_F\). Notice that \(\sigma \to 0\) as \(\tau \to 0\).

4. (Kittel 7.4) **Chemical potential versus temperature.** Explain graphically why the initial curvature of \(\mu\) versus \(\tau\) is upward for a fermion gas in one dimension and downward in three dimensions (Figure 7.7). Hint: The \(D_1(\varepsilon)\) and \(D_3(\varepsilon)\) curves are different, where \(D_1\) is given in Problem 1. It will be found useful to set up the integral for \(N\), the number of particles, and to consider from the graphs the behavior of the integrand between zero temperature and a finite temperature.

5. (Kittel 7.5) **Liquid \(^3\text{He} as a fermi gas.** The atom \(^3\text{He}\) has spin \(I = \frac{1}{2}\) and is a fermion.
   (a) Calculate as in Table 7.1 the fermi sphere parameters \(v_F\), \(\varepsilon_F\) and \(T_F\) for \(^3\text{He}\) at absolute zero, viewed as a gas of noninteracting fermions. The density of the liquid is 0.081 g cm\(^{-3}\).
   (b) Calculate the heat capacity at low temperatures \(T \ll T_F\) and compare with the experimental value \(C_V = 2.89 N k_B T\) as observed for \(T < 0.1 K\) by A. C. Anderson, W. Reese, and J. C. Wheatley, Phys. Rev. 130, 495 (1963); see also Figure 7.18.
   Excellent surveys of the properties of liquid \(^3\text{He}\) are given by J. Wilks, Properties of liquid and solid helium, Oxford, 1967, and by J. C. Wheatley, "Dilute solutions of \(^3\text{He}\) and \(^4\text{He}\) at low temperatures," American Journal of Physics, 36, 181-210 (1968). The principles of refrigerators based on \(^3\text{He} - ^4\text{He}\) are reviewed in Chapter 12 on cryogenics; such refrigerators produce steady temperatures down to 0.01K in continuously acting operation.

6. (Kittel 7.6) **Mass-radius relationship for white dwarfs.** Consider a white dwarf of mass \(M\) and radius \(R\). Let the electrons be degenerate but nonrelativistic.
   (a) Show that the order of magnitude of the gravitational self-energy is \(-GM^2/R\), where \(G\) is the gravitational constant. (If the mass density is constant within the sphere of radius \(R\), the exact potential energy is \(-3GM^2/5R\).)
   (b) Show that the order of magnitude of the kinetic energy of the electrons in the ground state is
\[
\frac{\hbar^2 N^{5/3}}{mR^2} \approx \frac{\hbar^2 M^{5/3}}{m(M_H)^{5/3} R^2}
\]

where \( m \) is the mass of an electron and \( M_H \) is the mass of a proton.

(c) Show that if the gravitational and kinetic energies are of the same order of magnitude (as required by the virial theorem of mechanics), \( M^{1/3} R \approx 10^{20} \text{ g}^{1/3} \text{ cm} \).

(d) If the mass is equal to that of the Sun \( (2 \times 10^{33} \text{ g}) \), what is the density of the white dwarf?

(e) It is believed that pulsars are stars composed of a cold degenerate gas of neutrons. Show that for a neutron star \( M^{1/3} R \approx 10^{17} \text{ g}^{1/3} \text{ cm} \). What is the value of the radius for a neutron star with a mass equal to that of the Sun? Express the result in km.