Homework #9

1. Frequency dependence of the electrical conductivity. Use the equation
   \[ m(dv/dt + v/\tau) = -eE \]
   for the electrical drift velocity \( v \) to show that the conductivity at frequency \( \omega \) is
   \[ \sigma(\omega) = \sigma(0) \left( \frac{1 + i\omega\tau}{1 + (\omega\tau)^2} \right), \]
   where \( \sigma(0) = ne^2\tau/m \).

   Solution:
   For \( E = E_0 e^{-i\omega t}, \ v = v_0 e^{-i\omega t} \). Then \( m(dv/dt + v/\tau) = -eE \) becomes
   \[ m(-i\omega v + v/\tau) = -eE, \ or \ v = -\frac{eE}{m(-i\omega + 1/\tau)} = -\frac{e\tau E}{m(1-i\omega\tau)}. \]
   The conductivity is
   \[ \sigma(\omega) = \frac{-nev}{E} = \frac{ne^2\tau}{m(1-i\omega\tau)} = \sigma(0) \left( \frac{1 + i\omega\tau}{1 + (\omega\tau)^2} \right). \]

2. Dynamic magnetoconductivity tensor for free electrons. A metal with a concentration \( n \) of free electrons of charge \(-e\) is in a static magnetic field \( B \hat{z} \). The electric current density in the \( xy \) plane is related to the electric field by
   \[ j_x = \sigma_{xx} E_x + \sigma_{xy} E_y; \quad j_y = \sigma_{yx} E_x + \sigma_{yy} E_y. \]
   Assume that the frequency \( \omega \gg \omega_c \) and \( \omega \gg 1/\tau \), where \( \omega_c = eB/mc \) and \( \tau \) is the collision time.
   (a) Solve the drift velocity equation
   \[ m(d\tilde{v}/dt + \tilde{v}/\tau) = -e(\tilde{E} + \tilde{B} \times \tilde{v}/c) \]
   to find the components of the magnetoconductivity tensor:
   \[ \sigma_{xx} = \sigma_{yy} = i\omega_p^2/4\pi\omega; \quad \sigma_{xy} = -\sigma_{yx} = \omega_c\omega_p^2/4\pi\omega^2, \]
   where \( \omega_p^2 = 4\pi ne^2/m \). (b) Note from the Maxwell equation that the dielectric function tensor of the medium is related to the conductivity tensor as \( \varepsilon = 1 + i(4\pi/\omega)\sigma \). Consider an electromagnetic wave with wavevector \( \tilde{k} = k\hat{z} \).
   Show that the dispersion relation for this wave in the medium is
   \[ c^2k^2 = \omega^2 - \omega_p^2 \pm \omega_c\omega_p^2/\omega. \]
   At a given frequency there are two modes of propagation with different wavevectors and different velocities. The two modes correspond to circularly polarized waves. Because a linearly polarized wave can be decomposed into two circularly polarized waves, it follows that the plane of polarization of a linearly polarized wave will be rotated by the magnetic field.

   Solution:
   (a) For \( E = E_0 e^{-i\omega t}, \ v = v_0 e^{-i\omega t} \). Then \( m(d\tilde{v}/dt + \tilde{v}/\tau) = -e(\tilde{E} + \tilde{v} \times \tilde{B}/c) \) becomes
\[ m(-i\omega \hat{v} + \hat{v} / \tau) = -e(\hat{E} + \hat{v} \times \hat{B} / c) \quad \text{or} \quad (1 - i\omega \tau)\hat{v} + \omega_c \hat{v} \times \hat{\zeta} = -\frac{e\tau}{m} \hat{E} \] where 
\[ \omega_c \equiv eB / mc \] is the cyclotron frequency.

The equation in x, y, and z directions are:
\[
\begin{align*}
(1 - i\omega \tau)v_x + \omega_c v_y &= -e\tau E_x / m \\
-\omega_c v_x + (1 - i\omega \tau)v_y &= -e\tau E_y / m \\
(1 - i\omega \tau)v_z &= -e\tau E_z / m
\end{align*}
\]

Solving these three equations gives the solution:
\[
\begin{align*}
v_x &= -\frac{e\tau[(1 - i\omega \tau)E_x - \omega_c \tau E_y]}{m[(1 - i\omega \tau)^2 + (\omega_c \tau)^2]} \\
v_y &= -\frac{e\tau[\omega_c \tau E_x + (1 - i\omega \tau)E_y]}{m[(1 - i\omega \tau)^2 + (\omega_c \tau)^2]} \\
v_z &= -\frac{e\tau E_z}{m(1 - i\omega \tau)}
\end{align*}
\]

With \( \vec{j} = -ne\hat{v} \), we have
\[
\begin{align*}
j_x &= \frac{ne^2\tau[(1 - i\omega \tau)E_x - \omega_c \tau E_y]}{m[(1 - i\omega \tau)^2 + (\omega_c \tau)^2]} \\
j_y &= \frac{ne^2\tau[\omega_c \tau E_x + (1 - i\omega \tau)E_y]}{m[(1 - i\omega \tau)^2 + (\omega_c \tau)^2]} \\
j_z &= \frac{ne^2\tau E_z}{m(1 - i\omega \tau)}
\end{align*}
\]

Then
\[
\begin{align*}
\sigma_{xx} &= \sigma_{yy} = \frac{ne^2\tau(1 - i\omega \tau)}{m[(1 - i\omega \tau)^2 + (\omega_c \tau)^2]} = \frac{\omega_p^2\tau(1 - i\omega \tau)}{4\pi[(1 - i\omega \tau)^2 + (\omega_c \tau)^2]} \\
\sigma_{xy} &= -\sigma_{yx} = -\frac{ne^2\omega_c \tau^2}{m[(1 - i\omega \tau)^2 + (\omega_c \tau)^2]} = -\frac{\omega_p^2\omega_c \tau^2}{4\pi[(1 - i\omega \tau)^2 + (\omega_c \tau)^2]} \\
\sigma_{zz} &= \frac{ne^2\tau}{m(1 - i\omega \tau)} = \frac{\omega_p^2 \tau}{4\pi(1 - i\omega \tau)}
\end{align*}
\]

Here \( \omega_p^2 = 4\pi ne^2 / m \) is the plasmon frequency.

With the condition of \( \omega \gg \omega_c \) and \( \omega \gg 1 / \tau \),
\[
\begin{align*}
\sigma_{xx} &= \sigma_{yy} \approx \frac{-\omega_p^2 \pi \omega \tau}{-4\pi(\omega \tau)^2} = \frac{i\omega_p^2}{4\pi \omega} \\
\sigma_{xy} &= -\sigma_{yx} \approx -\frac{\omega_p^2 \omega_c \tau^2}{-4\pi(\omega \tau)^2} = \frac{\omega_p^2 \omega_c}{4\pi \omega^2} \\
\sigma_{zz} &\approx \frac{i\omega_p^2}{4\pi \omega}
\end{align*}
\]
(b) \( \varepsilon = 1 + i(4\pi / \omega)\sigma \) gives the dielectric tensor of

\[
\varepsilon = \begin{pmatrix}
1 + \frac{4\pi i\sigma_{xx}}{\omega} & \frac{4\pi i\sigma_{xy}}{\omega} & 0 \\
\frac{4\pi i\sigma_{yx}}{\omega} & 1 + \frac{4\pi i\sigma_{yy}}{\omega} & 0 \\
0 & 0 & 1 + \frac{4\pi i\sigma_{zz}}{\omega}
\end{pmatrix} = \begin{pmatrix}
1 - \frac{\omega_p^2}{\omega^2} & \frac{i\omega_p^2\omega_c}{\omega^3} & 0 \\
\frac{i\omega_p^2\omega_c}{\omega^3} & 1 - \frac{\omega_p^2}{\omega^2} & 0 \\
0 & 0 & 1 - \frac{\omega_p^2}{\omega^2}
\end{pmatrix}
\]

For an EM wave with wavevector \( \vec{k} = k\hat{z} \), \( E_z = 0 \). The \( \vec{D} = e\vec{E} \) becomes

\[
\begin{pmatrix}
D_x \\
D_y
\end{pmatrix} = \begin{pmatrix}
1 - \frac{\omega_p^2}{\omega^2} & \frac{i\omega_p^2\omega_c}{\omega^3} \\
\frac{i\omega_p^2\omega_c}{\omega^3} & 1 - \frac{\omega_p^2}{\omega^2}
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_y
\end{pmatrix}
\]

The two eigen modes are determined by

\[
\det \begin{pmatrix}
\varepsilon - \left(1 - \frac{\omega_p^2}{\omega^2}\right) & -\frac{i\omega_p^2\omega_c}{\omega^3} \\
\frac{i\omega_p^2\omega_c}{\omega^3} & \varepsilon - \left(1 - \frac{\omega_p^2}{\omega^2}\right)
\end{pmatrix} = 0
\]

\[
\varepsilon - \left(1 - \frac{\omega_p^2}{\omega^2}\right)^2 - \left[\frac{\omega_p^2\omega_c}{\omega^3}\right]^2 = 0
\]

\[
\varepsilon = 1 - \frac{\omega_p^2}{\omega^2} \pm \frac{\omega_p^2\omega_c}{\omega^3}
\]

Note \( \omega = ck / \sqrt{\varepsilon} \). We then have \( c^2k^2 = \omega^2 \varepsilon = \omega^2 - \omega_p^2 \pm \omega_p \omega_c / \omega \)

For \( \varepsilon = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_p^2\omega_c}{\omega^3} \), solving the original equation gives \( E_y = iE_x \), \( D_y = iD_x \).

For \( \varepsilon = 1 + \frac{\omega_p^2}{\omega^2} + \frac{\omega_p^2\omega_c}{\omega^3} \), solving the original equation gives \( E_y = -iE_x \), \( D_y = -iD_x \).

These two modes then correspond to left and right circularly polarized light.

3. **Static magnetoconductivity tensor.** For the drift velocity theory of \( m(d\vec{v} / dt + \vec{v} / \tau) = -e(\vec{E} + \vec{v} \times \vec{B} / c) \), with \( \vec{B} = B\hat{z} \), show that the static current density can be written in matrix form as

\[
\begin{pmatrix}
\dot{j}_x \\
\dot{j}_y \\
\dot{j}_z
\end{pmatrix} = \frac{\sigma_0}{1 + (\omega_c \tau)^2} \begin{pmatrix}
1 & -\omega_c \tau & 0 \\
\omega_c \tau & 1 & 0 \\
0 & 0 & 1 + (\omega_c \tau)^2
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_y \\
E_z
\end{pmatrix}
\]

In the high magnetic field limit of \( \omega_c \tau >> 1 \), show that
\[ \sigma_{xx} = \frac{nec}{B} = -\sigma_{xy} \]

In this limit \( \sigma_{xx} = 0 \), to order \( 1/\omega_c \tau \). The quantity \( \sigma_{xy} \) is called the Hall conductivity.

**Solution:**
From the result of the last problem, we have

\[
\sigma_{xx} = \sigma_{yy} = \frac{\omega_p^2 \tau (1 - i \omega \tau)}{4\pi [(1 - i \omega \tau)^2 + (\omega_c \tau)^2]}
\]

\[
\sigma_{xy} = -\sigma_{yx} = -\frac{\omega_p^2 \omega_c \tau^2}{4\pi [(1 - i \omega \tau)^2 + (\omega_c \tau)^2]}
\]

\[
\sigma_{\infty} = \frac{\omega_p^2 \tau}{4\pi (1 - i \omega \tau)}
\]

At \( \omega = 0 \), the result is

\[
\sigma_{xx} = \sigma_{yy} = \frac{\omega_p^2 \tau}{4\pi [1 + (\omega_c \tau)^2]} = \frac{\sigma_0}{1 + (\omega_c \tau)^2}
\]

\[
\sigma_{xy} = -\sigma_{yx} = -\frac{\omega_p^2 \omega_c \tau^2}{4\pi [1 + (\omega_c \tau)^2]} = -\frac{\sigma_0 \omega_c \tau}{1 + (\omega_c \tau)^2}
\]

\[
\sigma_{\infty} = \frac{\omega_p^2 \tau}{4\pi} = \sigma_0
\]

Here \( \sigma_0 = \frac{\omega_p^2 \tau}{4\pi} = \frac{ne^2 \tau}{m} \).

Then the \( \vec{j} = \sigma \vec{E} \) can be written as

\[
\begin{pmatrix}
 j_x \\
 j_y \\
 j_z
\end{pmatrix} = \sigma_0 \begin{pmatrix}
 1 & -\omega_c \tau & 0 \\
 \omega_c \tau & 1 & 0 \\
 0 & 0 & 1 + (\omega_c \tau)^2
\end{pmatrix} \begin{pmatrix}
 E_x \\
 E_y \\
 E_z
\end{pmatrix}.
\]

In the high magnetic field limit of \( \omega_c \tau >> 1 \),

\[
\sigma_{xx} = \sigma_{yy} \approx \frac{\sigma_0}{(\omega_c \tau)^2}
\]

\[
\sigma_{xy} = -\sigma_{yx} \approx -\frac{\sigma_0 \omega_c \tau}{(\omega_c \tau)^2} = -\frac{\sigma_0}{\omega_c \tau} = -\frac{ne^2 \tau / m}{eB \tau / mc} = -\frac{nec}{B}
\]

4. **Maximum surface resistance.** Consider a square sheet of side \( L \), thickness \( d \), and electrical resistivity \( \rho \). The resistance measured between opposite edges of the sheet is called the surface resistance: \( R_{sq} = \rho L / Ld = \rho / d \), which is independent of the area \( L^2 \) of the sheet. \( R_{sq} \) is called the resistance per square and is expected in ohms per
square, because $\rho / d$ has the dimensions of ohms.) If we express $\rho$ by $\rho = m/n e^2 \tau$, then $R_{sq} = m/n e^2 \tau$. Suppose now that the minimum value of the collision time is determined by scattering from surfaces of the sheet, so that $\tau \approx d / v_F$, where $v_F$ is the Fermi velocity. Thus the maximum surface resistivity is $R_{sq} \approx m v_F / n d^2 e^2$. Show for a monatomic metal sheet one atom thickness that $R_{sq} \approx h / e^2 = 4.1k\Omega$, where 1kΩ is 10³ ohms.

Solution:
For 2D system,

$$N = \int_0^{e_F} DdE = \int_0^{e_F} 2 \frac{S2\pi dp}{h^2} = \int_0^{e_F} 2 \frac{S\pi m e_F}{h^2} = \frac{S4\pi m e_F}{h^2} = \frac{S4\pi m v_F^2}{2\pi h^2}$$

$$v_F = \frac{h}{m}\frac{2\pi N}{S}.$$

$$R_{sq} \approx m v_F / n d^2 e^2 = \frac{m}{m} \frac{h}{2\pi S} \frac{1}{n d^2 e^2} = \frac{h}{n d^2 e^2} \sqrt{\frac{2\pi N}{S}}.$$  

With $\sqrt{\frac{N}{S}} \sim \frac{1}{a}$, $n \sim \frac{1}{a^2}$, and $d \sim a$, we have

$$R_{sq} \sim \frac{h}{(1/a^3)a^2 e^2 a} = \frac{h}{e^2} = \frac{1.05 \times 10^{-34}}{(1.6 \times 10^{-19})^2} \Omega = 4.1 \times 10^3 \Omega = 4.1k\Omega.$$